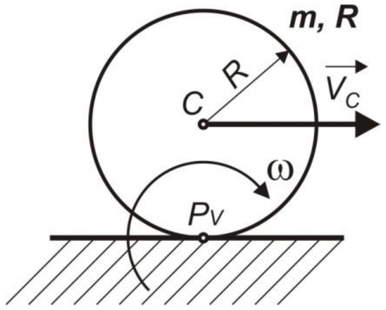
Kinetička energija krutog tela

Kinetička energija krutog tela predstavlja zbir kinetičke energije translacije (posledica translatornog kretanja tela) i kinetičku energiju rotacije tela oko ose (posledica rotacionog kretanja tela)

Primer: **kotrljanje kružnog diska**

Disk mase m i poluprečnika R može da se kotrlja bez klizanja po glatkoj horizontalnoj podlozi. Odrediti njegovu kinetičku enrgiju.



$$E_K = E_{Ktran.} + E_{Krot.} = \frac{1}{2}\, mV_C^2 + \frac{1}{2}\, J_c\omega_{aps}^2$$

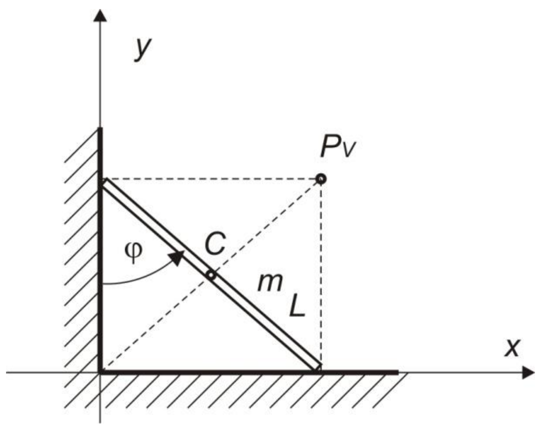
$\omega_{aps} = \frac{V_C}{R}$ apsolutna ugaona brzina oko centra

$J_C = \frac{1}{2}\, m\, R^2$ moment inercije diska za osu kroz centar upravnu na ravan diska

$$E_K = \frac{1}{2}\, mV_C^2 + \frac{1}{2}\frac{1}{2}\, m\, R^2 \left(\frac{V_C}{R}\right)^2 = \frac{1}{2}\, mV_C^2 + \frac{1}{4}\, m\, R^2 \frac{{V_C}^2}{R^2} = \frac{3}{4}\, mV_C^2$$

Primer: **štap dužine L i mase m**

Štap mase m i dužine L oslonjen je na glatki vertikalni zid i na glatki horizontalni pod. Odrediti njegovu kinetičku enrgiju.



$$E_K = E_{Ktran.} + E_{Krot.}$$

$$E_K = \frac{1}{2}\, mV_C^2 + \frac{1}{2}\, J_c\omega_{aps}^2$$

$\omega_{aps} = \dot{\varphi}$ apsolutna ugaona brzina oko centra

Preko trenutnog pola brzina

$$V_C = \frac{1}{2}L\dot{\varphi}$$

$$\overline{CP_V} = \sqrt{\left(\frac{1}{2}Lsin\varphi\right)^2 + \left(\frac{1}{2}Lcos\varphi\right)^2} = \frac{1}{2}L$$

Ili preko koordinata centra

$$x_C = \frac{1}{2}Lsin\varphi \qquad \dot{x}_C = \frac{1}{2}Lcos\varphi \cdot \dot{\varphi}$$

$$y_C = \frac{1}{2}Lcos\varphi \qquad \dot{y}_C = -\frac{1}{2}Lsin\varphi \cdot \dot{\varphi}$$

$$V_c = \sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{\left(\frac{1}{2}Lcos\varphi \cdot \dot{\varphi}\right)^2 + \left(-\frac{1}{2}Lsin\varphi \cdot \dot{\varphi}\right)^2} = \frac{1}{2}L \cdot \dot{\varphi}$$

$V_c = \frac{1}{2} L \cdot \dot{\varphi}$

$J_C = \frac{1}{12}\ m\ L^2$ moment inercije štapa za osu kroz centar upravnu na ravan diska

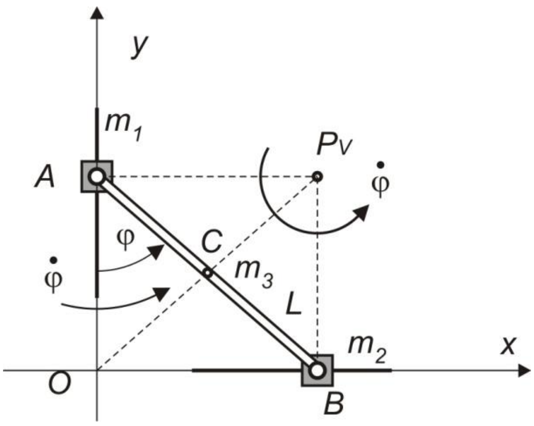
$$E_K = \frac{1}{2}\ mV_C^2 + \frac{1}{2}\frac{1}{12}\ m\ L^2(\dot{\varphi})^2 = \frac{1}{2}\ m\left(\frac{1}{2}\ L\dot{\varphi}\right)^2 + \frac{1}{2}\frac{1}{12}\ m\ L^2(\dot{\varphi})^2$$

$$E_K = \frac{1}{8}\ mL^2\dot{\varphi}^2 + \frac{1}{24}\ m\ L^2\dot{\varphi}^2 = \frac{4}{24}\ mL^2\dot{\varphi}^2 = \frac{1}{6}\ mL^2\dot{\varphi}^2$$

$$E_K = \frac{1}{6}\ mL^2\dot{\varphi}^2$$

Primer: **primer klizača povezanih štapom**

Odrediti kinetičku energiju sistema koji čine klizači A i B masa m_1 i m_2 zanemarljivih dimenzija i štapa AB dužine L i mase m_3.



Kinetička energija sistema jednaka je zbiru kinetičkih energija komponenata sistema

Kada je deo zanemarljivih dimenzija može se smatrati materijalnom tačkom, za koju je

$$E_K = \frac{1}{2}\ mV_C^2$$

Sistem ima dva klizača A i B i štap L

$$E_K = E_{KA} + E_{KB} + E_{KL}$$

$$E_{KA} = \frac{1}{2}\ m_1 V_A^2$$

$$E_{KB} = \frac{1}{2}\ m_2 V_B^2$$

$E_K = \frac{1}{6}\ m_3 L^2\dot{\varphi}^2$ kinetička energija za štap izvedena u prethodnom delu

$V_A = \dot{y}_A \quad y_A = Lcos\varphi \quad \dot{y}_A = -Lsin\varphi \cdot \dot{\varphi} = V_A$

$V_B = \dot{x}_B \quad x_B = Lsin\varphi \quad \dot{x}_B = Lcos\varphi \cdot \dot{\varphi} = V_B$

$$E_{KA} = \frac{1}{2}\ m_1 V_A^2 = \frac{1}{2}\ m_1(-Lsin\varphi \cdot \dot{\varphi})^2 = \frac{1}{2}\ m_1 L^2 sin^2\varphi \cdot \dot{\varphi}^2$$

$$E_{KB} = \frac{1}{2}\ m_2 V_B^2 = \frac{1}{2}\ m_2(Lcos\varphi \cdot \dot{\varphi})^2 = \frac{1}{2}\ m_2 L^2 cos^2\varphi \cdot \dot{\varphi}^2$$

$$E_K = \frac{1}{2}\ m_1 L^2 sin^2\varphi \cdot \dot{\varphi}^2 + \frac{1}{2}\ m_2 L^2 cos^2\varphi \cdot \dot{\varphi}^2 + \frac{1}{6}\ m_3 L^2\dot{\varphi}^2$$

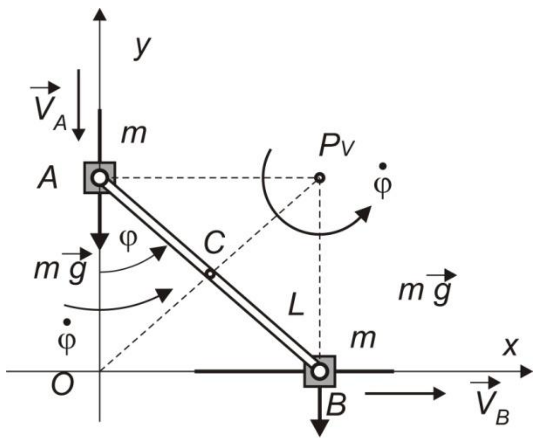
U slučaju da su mase jednake $m_1 = m_2 = m_3 = m$

$$E_K = E_{KA} + E_{KB} + E_{KL} = \frac{1}{2}\, mL^2\dot{\varphi}^2(sin^2\varphi + cos^2\varphi) + \frac{1}{6}\, mL^2\dot{\varphi}^2$$

$$E_K = \frac{2}{3}\, mL^2\dot{\varphi}^2$$

Zadatak 1.

Dva klizača A i B istih masa m spojeni su štapom dužine L zanemarljive mase i težine. Klizač A se kreće po vertikali a klizač B po horizontali. Zanemariti sile trenja i primenom zakona o promeni kinetičke energije odrediti ubrzanje štapa.



Rešenje:

Pošto je štap zanemarljive mase i težine samo dve komponente se računaju u kinetičkoj energiji sistema

$$E_K = E_{KA} + E_{KB}$$

$$E_{KA} = \frac{1}{2}\, mV_A^2$$

$$E_{KB} = \frac{1}{2}\, mV_B^2$$

$$V_A = \overline{AP_V} \cdot \dot{\varphi} = Lsin\varphi \cdot \dot{\varphi}$$

$$V_B = \overline{BP_V} \cdot \dot{\varphi} = Lcos\varphi \cdot \dot{\varphi}$$

$$E_K = \frac{1}{2}\, mV_A^2 + \frac{1}{2}\, mV_B^2 = \frac{1}{2}\, mL^2 sin^2\varphi \cdot \dot{\varphi}^2 + \frac{1}{2}\, mL^2 cos^2\varphi \cdot \dot{\varphi}^2$$

$$E_K = \frac{1}{2}\, mL^2\dot{\varphi}^2$$

Rad konzervativnih sila

$$dA(\vec{F}) = \vec{F} d\vec{r}$$

$$A(mg) = \pm mgh$$

$$A(mg) = -mg\, L\,(cos\varphi - cos\varphi_0)$$

$$E_K = A(m, \vec{g})$$

$$\frac{1}{2}\, mL^2\dot{\varphi}^2 = -mg\, L\,(cos\varphi - cos\varphi_0)$$

$$\frac{1}{2}\, L\dot{\varphi}^2 = -g\,(cos\varphi - cos\varphi_0)$$

$$\dot{\varphi} = \sqrt{\frac{g}{L}\ (cos\varphi_0 - cos\varphi)}$$

$$dE_K = dA$$

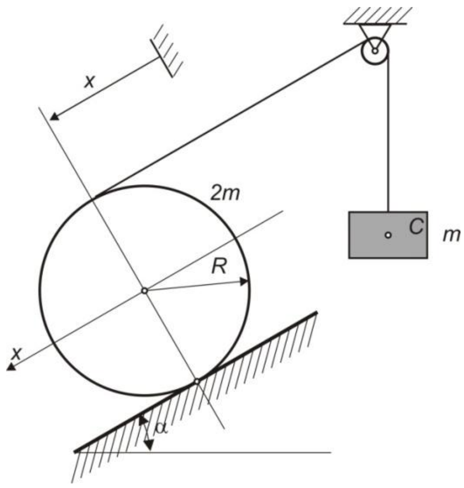
$$d\left(\frac{1}{2}\ mL^2\dot{\varphi}^2\right) = d\big(-mg\ L\ (cos\varphi - cos\varphi_0)\big)$$

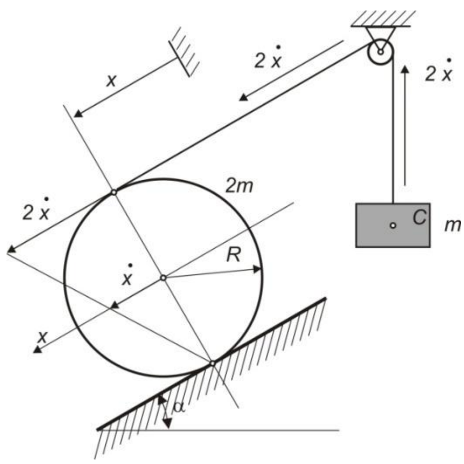
$$\frac{1}{2}\ mL^2 2\dot{\varphi}\ddot{\varphi} = -mg\ L\ (-sin\varphi)\dot{\varphi}$$

$$\ddot{\varphi} = \frac{g}{L}\ sin\varphi$$

Zadatak 2.

Homogeni kružni disk poluprečnika R i mase 2m, kotrlja se bez klizanja po strmoj ravni nagiba α. Oko diska je namotano uže koje je zatim prebačeno preko nepokretnog kotura K i vezano za telo C mase m. Napisati diferencijalnu jednačinu kretanja sistema i naći silu u užetu.





Rešenje:

Pošto je nepokretni kotur K zanemarljive mase i preko njega je prebačeno uže sa zanemarljivim trenjem, a uže je zanemarljive mase, samo dva elementa čine sistem

$$E_K = E_{KC} + E_{K1}$$

$$E_{KC} = \frac{1}{2}\ mV_C^2 = \frac{1}{2}\ m(2\dot{x})^2 = 2m\dot{x}^2$$

$$E_{K1} = E_{K1tran.} + E_{K1rot.}$$

$$E_{K1} = \frac{1}{2}\ 2mV_1^2 + \frac{1}{2}\ \frac{1}{2}\ 2m\ R^2\left(\frac{V_1}{R}\right)^2$$

$$E_{K1} = m\dot{x}^2 + \frac{1}{2}\ m\ R^2\frac{\dot{x}^2}{R^2} = \frac{3}{2}\ m\dot{x}^2$$

$$E_K = E_{K1} + E_{KC}$$

$$E_K = 2m\dot{x}^2 + \frac{3}{2}\ m\dot{x}^2 = \frac{7}{2}\ m\dot{x}^2$$

Rad konzervativnih sila

$$dA\big(\vec{F}\big) = \vec{F}d\vec{r}$$

$$A(mg) = 2mg\ xsin\alpha - mg2x = 2mgx(sin\alpha - 1)$$

$$E_K - E_{K0} = A$$

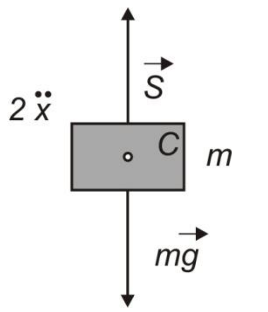
$$E_{K0} = 0$$

$$\frac{7}{2}\ m\dot{x}^2 = 2mgx(sin\alpha - 1)$$

da bi se odredilo ubrzanje potrebno je napraviti izvod po vremenu leve i desne strane

$\frac{7}{2}\ m\ 2\dot{x}\ \ddot{x} = 2mg\dot{x}(sin\alpha - 1)$

$\ddot{x} = \frac{2}{7}g(sin\alpha - 1) \rightarrow \ddot{x} < g$



Primeniti Dalamberov princip

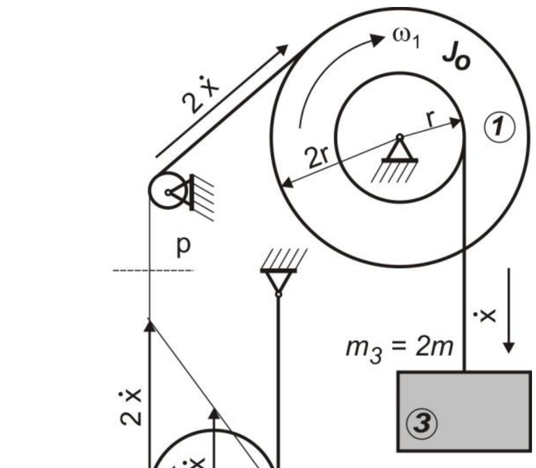
$m\ \vec{a} = m\vec{g} + \vec{S}$

$m2\ \ddot{x} = S - mg$

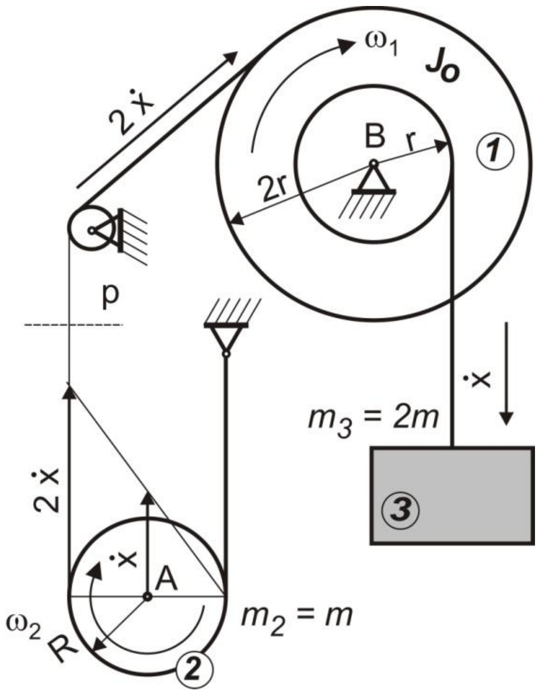
$S = mg + m2\ \ddot{x}\ = mg + \frac{4}{7}gm(sin\alpha - 1) = mg\left[1 + \frac{4}{7}(sin\alpha - 1)\right]$

Zadatak 3.

Materijalni sistem prikazan na slici sastoji se od kalema čiji je moment inercije J_0 kotura 2 mase m_2 i tereta 3 mase m_3.
Masu užadi kojom su povezana tela zanemariti.
Odrediti ubrzanje centra kotura 2 i silu u užetu u preseku p-p.



Rešenje:



Brzina kretanja kotura 2 i ugaona brzina kotura 2

$\dot{x} = R\omega_2 \rightarrow \omega_2 = \frac{\dot{x}}{R}$

Ugaona brzina kotura 1 $2\dot{x} = 2R\omega_2 = 2R\omega_1 \rightarrow \omega_1 = \omega_2 = \frac{\dot{x}}{R}$

Brzina spuštanja mase 3

$\dot{x}_3 = R\omega_1 = R\frac{\dot{x}}{R} = \dot{x}$

Kinetička energija sistema

$$E_K = E_{K1}+E_{K2}+E_{K3}$$

$$E_{K1} = \frac{1}{2} J_0\omega_1^2 = \frac{1}{2} J_0\frac{\dot{x}^2}{R^2} = \frac{1}{2} \frac{J_0}{R^2}\dot{x}^2$$

$$E_{K2} = \frac{1}{2} mV_2^2 + \frac{1}{4} m R^2\omega_2^2$$

$$E_{K2} = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} m R^2 \left(\frac{\dot{x}}{R}\right)^2 = m\dot{x}^2$$

$$E_{K3} = \frac{1}{2} 2mV_3^2 = \frac{1}{2} 2m\dot{x}^2 = m\dot{x}^2$$

$$E_K = E_{K1}+E_{K2}+E_{K2} = \frac{1}{2} \frac{J_0}{R^2}\dot{x}^2 + m\dot{x}^2 + m\dot{x}^2 = \dot{x}^2\left(\frac{J_0}{2R^2} + 2m\right)$$

Rad spoljašnjih sila

$$A = -mgx + 2mgx = mgx$$

$$\frac{dA}{dt} = mg\frac{dx}{dt} = mg\dot{x}$$

$$\frac{dE_K}{dt} = \left(\frac{J_0}{2R^2} + 2m\right)\frac{d(\dot{x}^2)}{dt} = \left(\frac{J_0}{2R^2} + 2m\right)2\dot{x}\ddot{x}$$

Primena zakona o održanju kinetičke energije

$$\frac{dE_K}{dt} = \frac{dA}{dt}$$

$$\left(\frac{J_0}{2R^2} + 2m\right)2\dot{x}\ddot{x} = mg\dot{x} \rightarrow \ddot{x} = \frac{mg}{2\left(\frac{J_0}{2R^2}+2m\right)} = \frac{mg\,R^2}{J_0+4R^2m}$$

Iz uslova ravnoteže određuju se sile u užadima

Ako se analizira ravnoteža kotura 2:

$$m\ddot{x} = -mg + S_1 + S_2$$

$$\frac{dL_A}{dt} = \sum M_A^{F_i} = S_2R - S_1R$$

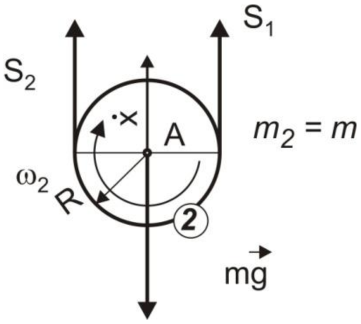
$$\vec{L}_A = \vec{M}_A^{\vec{K}} + J_c\vec{\omega}$$

projektovano na osu kroz A upravnu na ravan

$$K_2 = m\ddot{x} \rightarrow M_A^K = m\ddot{x} \cdot 0 = 0$$

$$J_A\omega_2 = \frac{1}{2}mR^2\frac{\dot{x}}{R} = \frac{1}{2}mR\dot{x}$$

$$L_A = \frac{1}{2}mR\dot{x} \rightarrow \frac{dL_A}{dt} = \frac{1}{2}mR\ddot{x}$$



$$\frac{dL_A}{dt} = \sum M_A^{F_i} = S_2 R - S_1 R$$

dobijen je sistem jednačina

$$\frac{1}{2} mR\ddot{x} = S_2 R - S_1 R$$

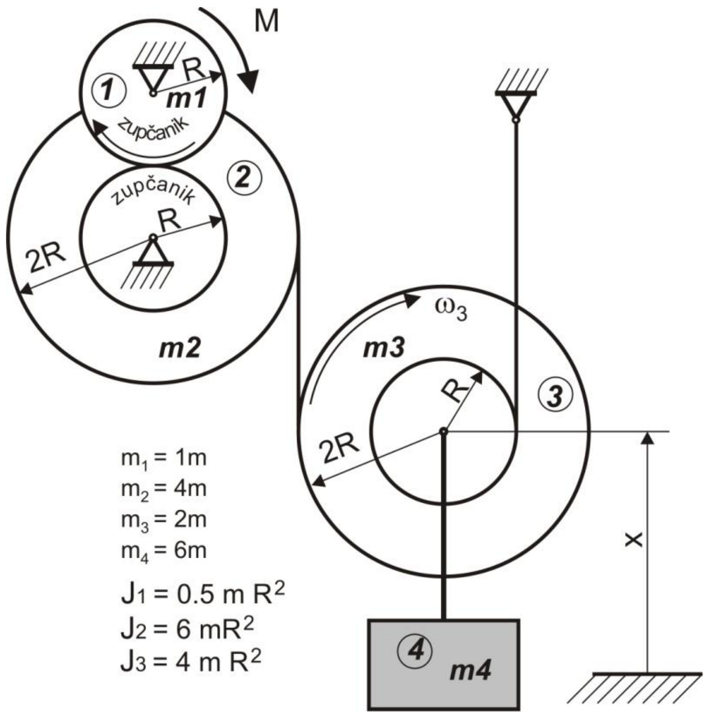
$$\frac{1}{2} m\ddot{x} = S_2 - S_1 \rightarrow S_1 = -\frac{1}{2} m\ddot{x} + S_2$$

$$m\ddot{x} = -mg - \frac{1}{2} m\ddot{x} + S_2 + S_2 \rightarrow 2S_2 = m\ddot{x} + mg + \frac{1}{2} m\ddot{x}$$

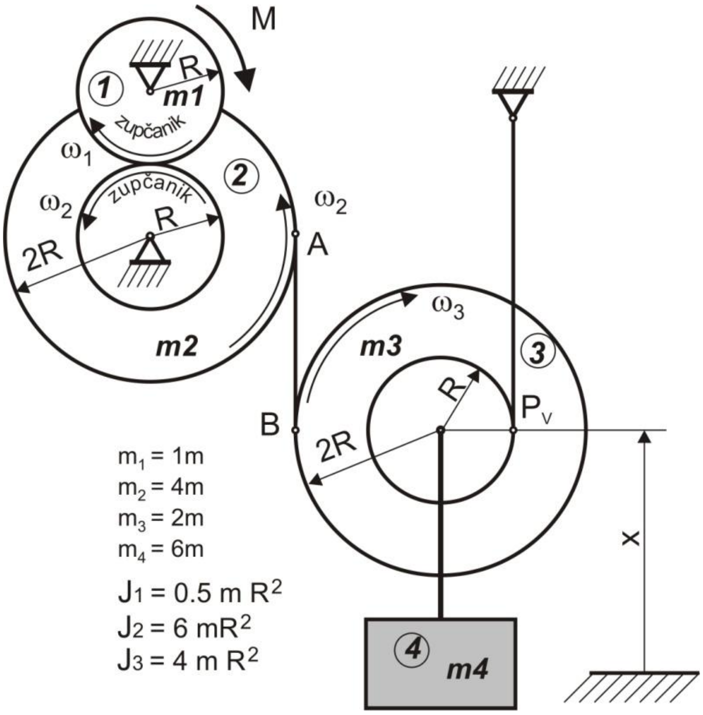
$$S_2 = m\left(g + \frac{3}{2}\ddot{x}\right) = m\left(g + \frac{3}{2}\frac{mg\,R^2}{J_0 + 4R^2 m}\right)$$

Zadatak 4.

Materijalni sistem prikazan na slici sastoji se od zupčanika 1 mase m i momenta inercije $J_2 = 0.5mR^2$, zavarenog zupčanika za kotur 2 ukupne mase 4m i momenta inercije za osu $J_2 = 6mR^2$. Preko kotura 2 i većeg prečnika kotura 3 je prebačeno lako uže koje se namotava na disk 2. Na manjem prečniku kotura 3 namotava se lako uže koje je drugim krajem zakačeno za nepomičnu tačku. Masa diska 3 je $m_3 = 2m$, a moment inercije $J_3 = 4mR^2$ Za osovinu kotura 3 zakačen je teret 4 mase $m_4 = 6m$. Zupčanik se pogoni momentom M i podiže teret. Odrediti ubzanje centra kotura 3.



Rešenje:



$$R\omega_1 = R\omega_2 \rightarrow \omega_1 = \omega_2$$

$$V_A = 2R\omega_2 = V_B$$

$$V_B = 3R\omega_3 = 2R\omega_2 \rightarrow \omega_3$$

$$\omega_3 = \frac{2}{3}\omega_2 = \frac{2}{3}\omega_1$$

$$V_3 = R\omega_3 = \frac{2}{3}R\dot{\varphi}$$

$$\varphi = \int \omega_1 dt$$

$$x = R\varphi_3 = \frac{2}{3}R\varphi$$

$$E_K = E_{K1}+E_{K2}+E_{K3}+E_{K4}$$

$$E_{K1} = \frac{1}{2}\,J_1\omega_1^2 = \frac{1}{2}\,\frac{1}{2}m\,R^2\dot{\varphi}^2$$

$$E_{K1} = \frac{1}{4}m\,R^2\dot{\varphi}^2$$

$$E_{K2} = \frac{1}{2}\,J_2\omega_2^2 = \frac{1}{2}\;6m\,R^2\dot{\varphi}_2^2$$

$$E_{K2} = 3m\,R^2\dot{\varphi}^2$$

$$E_{K3} = \frac{1}{2}\,2mV_3^2 + \frac{1}{2}\;4m\,R^2\omega_3^2 = m\frac{4}{9}R^2\dot{\varphi}^2 + 2m\frac{4}{9}R^2\dot{\varphi}^2 = \frac{12}{9}mR^2\dot{\varphi}^2$$

$$E_{K4} = \frac{1}{2}\;6mV_3^2 = 3m\frac{4}{9}R^2\dot{\varphi}^2 = \frac{12}{9}mR^2\dot{\varphi}^2 = \frac{4}{3}mR^2\dot{\varphi}^2$$

$$E_K = E_{K1}+E_{K2}+E_{K3}+E_{K4}$$

$$E_K = \frac{1}{4}m\,R^2\dot{\varphi}^2 + 3m\,R^2\dot{\varphi}^2 + \frac{4}{3}mR^2\dot{\varphi}^2 + \frac{4}{3}mR^2\dot{\varphi}^2 = \frac{71}{12}mR^2\dot{\varphi}^2$$

$$\frac{dE_K}{dt} = \frac{71}{12}mR^2 2\ddot{\varphi}\dot{\varphi} = \frac{71}{6}mR^2\ddot{\varphi}\dot{\varphi}$$

$$A = -6mgx + M\varphi = -6mg\frac{2}{3}R\varphi + M\varphi = (M - 4mgR)\varphi$$

$$\frac{dA}{dt} = (M - 4mg)\dot{\varphi}$$

$$\frac{dE_K}{dt} = \frac{dA}{dt}$$

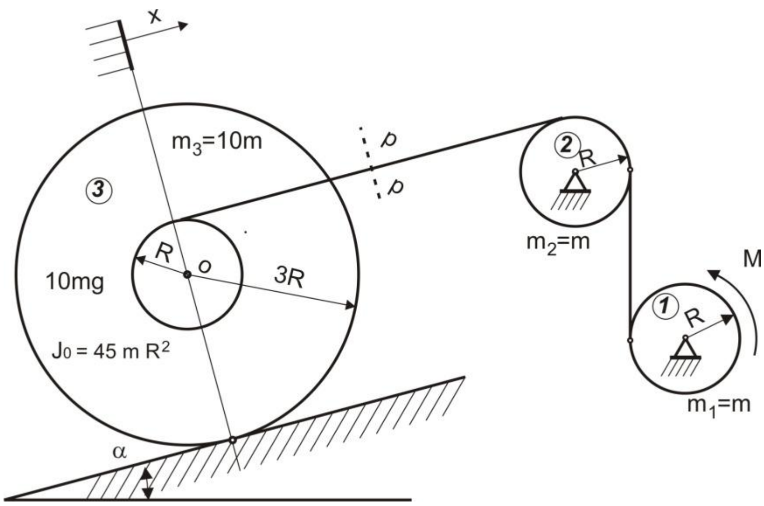
$$\frac{71}{6} mR^2 \ddot{\varphi}\dot{\varphi} = (M - 4mg)\dot{\varphi}$$

$$\ddot{\varphi} = \frac{6(M-4mg)}{71mR} \qquad x = R\varphi_3 = \frac{2}{3}R\varphi$$

$$\ddot{x} = R\ddot{\varphi}_3 = \frac{2}{3}R\ddot{\varphi} = \frac{2}{3}R\frac{6(M-4mg)}{71mR} = \frac{4(M-4mg)}{71mR}$$

Zadatak 5.

Materijalni sistem prikazan na slici sastoji se od diska 1, kotura 2, kalema 3 i lakog užeta koje se namotava na disk 1, a prebačeno je preko kotura 2 i namotano na doboš koji je deo kalema. Ako na disk dejstvuje spreg M odrediti ubrzanje centra O kalema koji se kotrlja bez klizanja po strmoj ravni nagiba α. Odrediti silu u užetu u naznačenom preseku p-p, ako je koeficijent trenja kalema 3 i podloge μ. Ostale podatke uzeti sa slike.

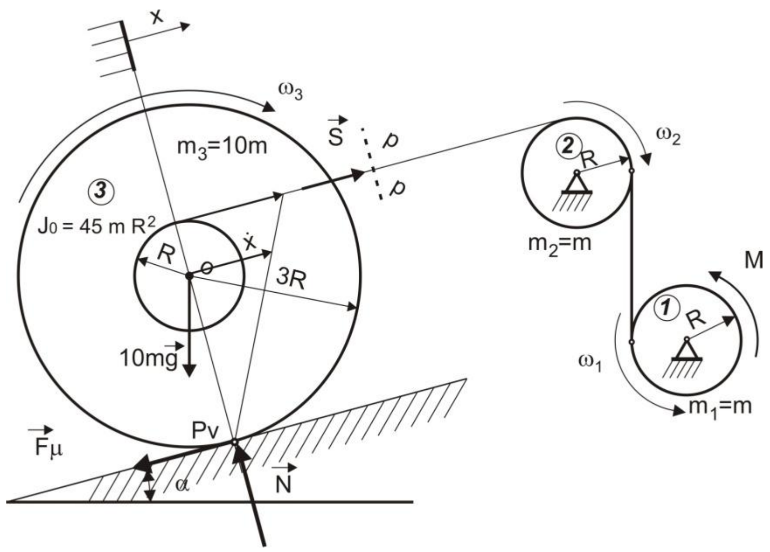


Rešenje:

$$R\omega_1 = R\omega_2 \rightarrow \omega_1 = \omega_2$$

$$V_A = 4R\omega_3 = R\omega_2 \rightarrow \omega_3$$

$$\omega_3 = \frac{1}{4}\omega_2 = \frac{1}{4}\omega_1$$



$\dot{x} = 3R\omega_3 = \frac{3R}{4}\omega_1$

$\varphi = \int \omega_1 dt$

$V_3 = 3R\omega_3 = \frac{3}{4}R\dot{\varphi}$

$x = 3R\varphi_3 = \frac{3}{4}R\varphi$

$E_K = E_{K1}+E_{K2}+E_{K3}$

$E_{K1} = \frac{1}{2}\frac{1}{2}\ mR^2\omega_1^2 = \frac{1}{4}\ mR^2\dot{\varphi}^2$

$E_{K2} = \frac{1}{2}\frac{1}{2}\ mR^2\omega_2^2 = \frac{1}{4}\ mR^2\dot{\varphi}^2$

$E_{K3} = \frac{1}{2}\ 10mV_o^2 + \frac{1}{2}J_o\omega_3^2 = \frac{1}{2}\ 10m(\dot{x})^2 + \frac{1}{2}45mR^2\omega_3^2$

$E_{K3} = \frac{45}{16}\ mR^2\dot{\varphi}^2 + \frac{45}{32}mR^2\dot{\varphi}^2 = \frac{135}{32}mR^2\dot{\varphi}^2$

$E_K = E_{K1}+E_{K2}+E_{K3}$

$$E_K = \frac{1}{4}\ mR^2\dot{\varphi}^2 + \frac{1}{4}\ mR^2\dot{\varphi}^2 + \frac{135}{32}mR^2\dot{\varphi}^2 = \frac{151}{32}\ mR^2\dot{\varphi}^2$$

$$\frac{dE_K}{dt} = \frac{151}{32}mR^2 2\ddot{\varphi}\dot{\varphi} = \frac{151}{16}mR^2\ddot{\varphi}\dot{\varphi}$$

$A = -10mgsin\alpha x + M\varphi = \left(M - 10mgsin\alpha\frac{3}{4}R\right)\varphi$

$\frac{dA}{dt} = \left(M - \frac{30}{4}mgsin\alpha R\right)\dot{\varphi}$

$$\frac{dE_K}{dt} = \frac{dA}{dt}$$

$$\frac{151}{16} mR^2 \ddot{\varphi}\dot{\varphi} = \left(M - \frac{30}{4} mgsin\alpha R\right)\dot{\varphi}$$

$$\ddot{\varphi} = \frac{16\left(M - \frac{30}{4} mgsin\alpha R\ \right)}{151 mR^2} = \frac{16}{151}\left(\frac{M}{mR^2} - \frac{30}{4R} gsin\alpha\right)$$

$$\ddot{x} = 3R\ddot{\varphi}_3 = \frac{3R}{4}\ddot{\varphi} = \frac{3R}{4}\frac{16}{151}\left(\frac{M}{mR^2} - \frac{30}{4R} gsin\alpha\right) = \frac{12}{151}\left(\frac{M}{mR} - \frac{15}{2} gsin\alpha\right)$$

Iz uslova ravnoteže kalema 3 određuje se sila u užetu:

$$10m\,\vec{a} = 10m\vec{g} + \vec{S}$$

$$10m\ddot{x} = -10mgsin\alpha + S - F_\mu$$

$$0 = -10mgcos\alpha + N \ \rightarrow N = 10mgcos\alpha \rightarrow F_\mu \ = \mu 10mgcos\alpha$$

$$10m\ddot{x} = -10mgsin\alpha + S - \mu 10mgcos\alpha$$

$$S = 10m\ddot{x} + 10mgsin\alpha + \mu 10mgcos\alpha = 10m[\ddot{x} + g(sin\alpha + \mu gcos\alpha)]$$

$$S = \frac{120}{151}\left(\frac{M}{R} - \frac{15}{2} mgsin\alpha\right) + 10mg(sin\alpha + \mu gcos\alpha)$$